Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Specimen exam 2

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

(a) We have $|1-i| = \sqrt{2}$ and $\operatorname{Arg}(1-i) = -\pi/4$, so $1-i = \sqrt{2}e^{-i\pi/4}$. 2

(b) From part (a), $1-i = \sqrt{2}e^{-i\pi/4}$, so

$$(1-i)^6 = (\sqrt{2}e^{-i\pi/4})^6 = 8e^{-6i\pi/4} = 8e^{-3i\pi/2} = 8e^{i\pi/2}.$$ 2

(c) We have $\operatorname{Log} i = \log 1 + i\pi/2 = i\pi/2$. Hence

$$i^{1-i} = e^{(1-i)\operatorname{Log} i} = e^{(1-i)i\pi/2} = e^{\pi/2}e^{i\pi/2}.$$ 3

(d) We have $|\sqrt{3}+i| = 2$ and $\operatorname{Arg}(\sqrt{3}+i) = \pi/6$, so $\sqrt{3}+i = 2e^{i\pi/6}$.

Also, $1-i = \sqrt{2}e^{-i\pi/4}$, by part (a).

Hence

$$\frac{\sqrt{3}+i}{1-i} = \frac{2e^{i\pi/6}}{\sqrt{2}e^{-i\pi/4}} = \sqrt{2}e^{5i\pi/12}.$$ 3

10 Total

Question 2

(a) (i) Let $f(z) = \dfrac{\cos z}{z-3}$.

This function is analytic on the simply connected region $\mathcal{R} = \{z : |z| < 3\}$, which contains the circle $C = \{z : |z| = 1\}$. It follows from Cauchy's Theorem that

$$\int_C \frac{\cos z}{z-3}\,dz = 0.$$ 2

(ii) We have seen that f is analytic on the simply connected region $\mathcal{R}$, and C lies in $\mathcal{R}$. Since 0 lies inside C, we can apply Cauchy's Integral Formula to give

$$\int_C \frac{\cos z}{z(z-3)}\,dz = \int_C \frac{f(z)}{z}\,dz = 2\pi i f(0) = -\frac{2\pi i}{3}.$$ 2

(iii) As before, we have that f is analytic on the simply connected region $\mathcal{R}$, and C lies inside $\mathcal{R}$. Since 0 lies inside C, we can apply Cauchy's Second Derivative Formula to give

$$\int_C \frac{\cos z}{z^3(z-3)}\,dz = \int_C \frac{f(z)}{z^3}\,dz = \frac{2\pi i}{2!}f''(0) = \pi i f''(0).$$

Observe that

$$f'(z) = \frac{(z-3)(-\sin z) - \cos z}{(z-3)^2} = -\frac{\sin z}{z-3} - \frac{\cos z}{(z-3)^2}.$$

Hence

$$f''(z) = -\frac{(z-3)\cos z - \sin z}{(z-3)^2} - \frac{(z-3)^2(-\sin z) - 2(z-3)\cos z}{(z-3)^4}.$$

So

$$f''(0) = -\frac{-3}{(-3)^2} - \frac{6}{(-3)^4} = \frac{1}{3} - \frac{2}{27} = \frac{7}{27}.$$

It follows that

$$\int_C \frac{\cos z}{z^3(z-3)}\,dz = \pi i f''(0) = \frac{7\pi i}{27}.$$ 4

(b) The contour Γ traverses the unit circle twice anticlockwise. By convention, the contour C traverses the unit circle once anticlockwise. Hence $\Gamma = C + C$. If g is a function that is continuous on Γ, then

$$\int_\Gamma g = \int_C g + \int_C g = 2\int_C g.$$

Therefore replacing C by Γ has the effect of doubling all the answers obtained in part (a). 2

10 Total

Question 3

(a) The function f has a simple pole at each of the four solutions ± 1 and $\pm i$ of $z^4 - 1 = 0$.

We can calculate the residue at each pole using the g/h Rule with $g(z) = z^2 + 2$ and $h(z) = z^4 - 1$, observing that $h'(z) = 4z^3$ is non-zero at each of the four poles of f. We obtain

$$\operatorname{Res}(f, 1) = \frac{g(1)}{h'(1)} = \frac{3}{4},$$

$$\operatorname{Res}(f, -1) = \frac{g(-1)}{h'(-1)} = -\frac{3}{4},$$

$$\operatorname{Res}(f, i) = \frac{g(i)}{h'(i)} = \frac{1}{-4i} = \frac{i}{4},$$

$$\operatorname{Res}(f, -i) = \frac{g(-i)}{h'(-i)} = \frac{1}{4i} = -\frac{i}{4}.$$ 4

(b) The function f is analytic on the simply connected region $\mathbb{C}$ apart from simple poles at ± 1 and $\pm i$, all of which lie inside Γ. Applying the Residue Theorem with the residues found in part (a) gives

$$\int_\Gamma \frac{z^2+2}{z^4-1}\,dz = 2\pi i\left(\frac{3}{4} - \frac{3}{4} + \frac{i}{4} - \frac{i}{4}\right) = 0.$$ 2

(c) Let $p(t) = t^2 + 2$ and $q(t) = t^4 - 1$. Then the degree of q exceeds that of p by $4 - 2 = 2$ and, by part (a), the poles of $f = p/q$ on the real axis are simple. Hence we can apply HB C1 3.8, p62, to see that

$$\int_{-\infty}^{\infty} \frac{t^2+2}{t^4-1}\,dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of f at the poles in the upper half-plane, and T is the sum of the residues of f at the poles on the real axis. Using the residues found in part (a) we see that

$$\int_{-\infty}^{\infty} \frac{t^2+2}{t^4-1}\,dt = 2\pi i \times \frac{i}{4} + \pi i\left(\frac{3}{4} - \frac{3}{4}\right) = -\frac{\pi}{2}.$$ 4

10 Total

Question 4

(a) By HB C3 3.8, p76, $\beta = -1$ (the centre of C) is the unique inverse point of ∞ with respect to C. 2

(b) We apply the inverse points method for finding images of generalised circles under Möbius transformations. Observe that

$$f(-1) = \frac{2}{-1-i} = -\frac{2(1-i)}{(1+i)(1-i)} = -1+i \quad \text{and} \quad f(\infty) = 0.$$

It follows that $-1+i$ and 0 are inverse points with respect to $f(C)$, so $f(C)$ has an equation of the form

$$|z+1-i| = k|z|, \quad \text{for some } k > 0.$$

Now, $0 \in C$, so

$$f(0) = \frac{2}{-i} = 2i \in f(C).$$

Hence

$$k = \frac{|2i+1-i|}{|2i|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

Therefore $f(C)$ has equation

$$|z| = \sqrt{2}|z+1-i|$$

in Apollonian form. 5

(c) By HB C3 3.11, p76, with $\alpha = 0$, $\beta = -1+i$ and $k = \sqrt{2}$, the centre of $f(C)$ is

$$\lambda = \frac{0 - (\sqrt{2})^2(-1+i)}{1-(\sqrt{2})^2} = -2+2i$$

and the radius is

$$r = \frac{\sqrt{2}|0-(-1+i)|}{|1-(\sqrt{2})^2|} = 2.$$ 3

10 Total

Question 5

(a) The conjugate velocity function

$$\overline{q}(z) = -\frac{i}{z^2}$$

is analytic on $\mathbb{C} - \{0\}$, so q is the velocity function for an ideal flow on $\mathbb{C} - \{0\}$, by HB D1 1.15, p81. The functions q and $\overline{q}$ are not defined at 0, so $\mathbb{C} - \{0\}$ is the largest region on which q represents an ideal flow. 2

(b) A complex potential function for the flow is

$$\Omega(z) = \frac{i}{z},$$

because this function is a primitive of $\overline{q}$ on $\mathbb{C} - \{0\}$. Writing $z = x + iy$, we see that

$$\Omega(z) = \frac{i\overline{z}}{|z|^2} = \frac{i(x - iy)}{x^2 + y^2} = \frac{y + ix}{x^2 + y^2}.$$

Hence a stream function for the flow is

$$\Psi(z) = \operatorname{Im}\Omega(z) = \frac{x}{x^2 + y^2}.$$

The streamlines are given by $\Psi(z) = k$, for real constants k. When $k = 0$ we obtain the equation

$$x = 0.$$

When $k \neq 0$, we obtain the equation $x^2 + y^2 - x/k = 0$. We can rewrite this equation as

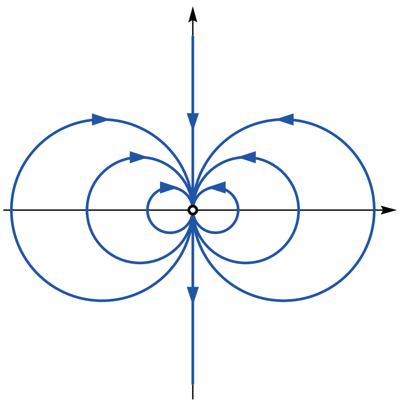
$$(x - c)^2 + y^2 = c^2,$$

where $c = 1/(2k)$. The origin (given by $x = y = 0$) is excluded from the equation of every streamline, because it is not in the flow region. 4

(c) The equation $x = 0$ represents two streamlines: the positive and negative imaginary axes. At the point i we have

$$q(i) = \frac{i}{(-i)^2} = -i.$$

So the flow is downwards on the positive imaginary axis. Using the continuity of q we can determine the direction of flow at all other points.



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10 Total

Question 6

(a) (i) The fixed points of f are the solutions of the fixed point equation

$$z^2 - z = z.$$

This can be rearranged to give $z^2 - 2z = 0$, which has solutions $z = 0$ and $z = 2$.

We have $f'(z) = 2z - 1$. Hence

$$|f'(0)| = |-1| = 1,$$

so 0 is an indifferent fixed point. Also,

$$|f'(2)| = 3,$$

so 2 is a repelling fixed point. 5

(ii) Any periodic point z of f of period 2 must satisfy the equation

$$f^2(z) = f(f(z)) = z.$$

Now,

$$f(f(z)) = (z^2 - z)^2 - (z^2 - z) = z^4 - 2z^3 + z.$$

Therefore $f^2(z) = z$ if and only if $z^4 - 2z^3 + z = z$, which is equivalent to the equation

$$z^3(z - 2) = 0.$$

The only solutions of this equation are $z = 0$ and $z = 2$. However, these are fixed points of f, so they are not periodic points of period 2. It follows that f has no periodic points of period 2 after all. 3

(b) By HB D2 4.7(c), p92, the Mandelbrot set M meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$. Since $-2 < -\sqrt{3} < \frac{1}{4}$, we see that $-\sqrt{3} \in M$. 2

10 Total

Question 7

(a) (i) Let $z = x + iy$. Then $f(z) = u(x, y) + iv(x, y)$, where

$$u(x, y) = e^y \cos x \quad \text{and} \quad v(x, y) = e^y \sin x.$$

The partial derivatives of u and v with respect to x and y are

$$\frac{\partial u}{\partial x}(x, y) = -e^y \sin x,$$

$$\frac{\partial u}{\partial y}(x, y) = e^y \cos x,$$

$$\frac{\partial v}{\partial x}(x, y) = e^y \cos x,$$

$$\frac{\partial v}{\partial y}(x, y) = e^y \sin x.$$

The first Cauchy–Riemann equation is

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \iff -e^y \sin x = e^y \sin x \iff e^y \sin x = 0.$$

Since $e^y \neq 0$, this equation is equivalent to the equation $\sin x = 0$, which has solutions $x = n\pi$, for $n \in \mathbb{Z}$.
The second Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \iff e^y \cos x = -e^y \cos x \iff e^y \cos x = 0.$$

Since $e^y \neq 0$, this equation is equivalent to the equation $\cos x = 0$, which has solutions $x = \left(n + \frac{1}{2}\right)\pi$, for $n \in \mathbb{Z}$.
Hence there are no points at which both the Cauchy–Riemann equations are satisfied. It follows from the Cauchy–Riemann Theorem that there are no points of $\mathbb{C}$ at which f is differentiable. 7

(ii) Let $z = x + iy$. Then

$$g(z) = g(x+iy) = e^x(\cos y + i \sin y) = e^z.$$

The exponential function is entire, so g is differentiable at every point in the complex plane. 2

(b) (i) We have

$$\gamma_1(\pi/2) = e^{i\pi/2} = i \quad \text{and} \quad \pi/2 \in [0, 2\pi],$$

$$\gamma_2(1) = 0 + i = i \quad \text{and} \quad 1 \in \mathbb{R}.$$

Hence Γ_1 and Γ_2 meet at the point i.
To work out the angle θ from Γ_1 to Γ_2 at i, we observe that

$$\gamma_1'(t) = ie^{it}, \quad \text{so} \quad \gamma_1'(\pi/2) = ie^{i\pi/2} = i^2 = -1$$

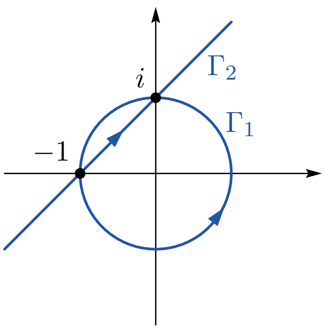
and

$$\gamma_2'(t) = 1 + i, \quad \text{so} \quad \gamma_2'(1) = 1 + i.$$

Hence

$$\theta = \operatorname{Arg}\left(\frac{\gamma_2'(1)}{\gamma_1'(\pi/2)}\right) = \operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}.$$ 4

(ii)



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(iii) We have

$$g'(z) = 2z \exp(z^2).$$

Since $\exp(z^2) \neq 0$, we see that $g'(z) = 0$ if and only if $z = 0$. It follows from HB A4 4.7, p40, that g is conformal on $\mathbb{C} - \{0\}$ and it is not conformal at 0. 2

(iv) The function g is conformal at i, so the angle from $g(\Gamma_1)$ to $g(\Gamma_2)$ at $g(i)$ is equal to the angle θ from Γ_1 to Γ_2 at i, which we found in part (b)(i) to be $\theta = -3\pi/4$. 2

20 Total

Question 8

(a) We have

$$\exp w = 1 + w + \frac{1}{2}w^2 + \frac{1}{6}w^3 + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Using the binomial series, we see that

$$\begin{aligned}(1+z)^{-1/3} &= 1 + \frac{(-1/3)}{1!}z + \frac{(-1/3)(-4/3)}{2!}z^2 + \frac{(-1/3)(-4/3)(-7/3)}{3!}z^3 + \cdots \\ &= 1 - \frac{1}{3}z + \frac{2}{9}z^2 - \frac{14}{81}z^3 + \cdots,\end{aligned}$$

for $|z| < 1$. Hence

$$(1+z)^{-1/3} - 1 = -\frac{1}{3}z + \frac{2}{9}z^2 - \frac{14}{81}z^3 + \cdots, \quad \text{for } |z| < 1.$$

Let $w = (1+z)^{-1/3} - 1$. Since $(1+0)^{-1/3} - 1 = 0$, we can apply the Composition Rule for Power Series to give

$$\begin{aligned}&\exp((1+z)^{-1/3} - 1) \\ &= 1 + \left(-\frac{1}{3}z + \frac{2}{9}z^2 - \frac{14}{81}z^3 + \cdots\right) + \frac{1}{2}\left(-\frac{1}{3}z + \frac{2}{9}z^2 - \cdots\right)^2 + \frac{1}{6}\left(-\frac{1}{3}z + \cdots\right)^3 + \cdots \\ &= 1 + \left(-\frac{1}{3}z + \frac{2}{9}z^2 - \frac{14}{81}z^3 + \cdots\right) + \frac{1}{2}\left(\frac{1}{9}z^2 - \frac{4}{27}z^3 + \cdots\right) + \frac{1}{6}\left(-\frac{1}{27}z^3 + \cdots\right) + \cdots \\ &= 1 - \frac{1}{3}z + \frac{5}{18}z^2 - \frac{41}{162}z^3 + \cdots,\end{aligned}$$

for $|z| < r$, where r is some positive constant. 7

(b) Using partial fractions we can write

$$\frac{5}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2},$$

for constants A and B. Multiplying both sides by $(z-3)(z+2)$, we obtain

$$5 = A(z+2) + B(z-3).$$

Setting $z = 3$ gives $5 = 5A$, so $A = 1$. Setting $z = -2$ gives $5 = -5B$, so $B = -1$. Hence

$$\frac{5}{(z-3)(z+2)} = \frac{1}{z-3} - \frac{1}{z+2}.$$

(Check: when $z = 0$, the LHS is $-\frac{5}{6}$ and the RHS is $-\frac{1}{3} - \frac{1}{2} = -\frac{5}{6}$.)

Let $w = z + 1$. Then $z = w - 1$, so

$$g(z) = \frac{1}{z-3} - \frac{1}{z+2} = \frac{1}{w-4} - \frac{1}{w+1}.$$

If $1 < |z+1| < 4$, then $1 < |w| < 4$, so

$$g(z) = \frac{1}{w-4} - \frac{1}{w+1} = -\frac{1}{4} \times \frac{1}{1 - w/4} - \frac{1}{w} \times \frac{1}{1 + 1/w},$$

where $|w/4| < 1$ and $|1/w| < 1$.

Hence

$$\begin{aligned}
g(z) &= -\frac{1}{4}\left(1+\frac{w}{4}+\left(\frac{w}{4}\right)^2+\cdots\right)-\frac{1}{w}\left(1-\frac{1}{w}+\left(\frac{1}{w}\right)^2-\cdots\right)\\
&= \left(-\frac{1}{4}-\frac{w}{4^2}-\frac{w^2}{4^3}-\cdots\right)-\left(\frac{1}{w}-\frac{1}{w^2}+\frac{1}{w^3}-\cdots\right)\\
&= \cdots+\frac{1}{w^2}-\frac{1}{w}-\frac{1}{4}-\frac{w}{4^2}-\frac{w^2}{4^3}-\cdots\\
&= \cdots+\frac{1}{(z+1)^2}-\frac{1}{(z+1)}-\frac{1}{4}-\frac{(z+1)}{4^2}-\frac{(z+1)^2}{4^3}-\cdots\\
&= \cdots+\frac{1}{(z+1)^2}-\frac{1}{(z+1)}-\frac{1}{4}-\frac{(z+1)}{16}-\frac{(z+1)^2}{64}-\cdots,
\end{aligned}$$

for $1 < |z+1| < 4$. 10

(c) The functions f_1 and f_2 are analytic on the region $\mathbb{C}$ and they agree on the subset $\mathbb{Q}$ of $\mathbb{C}$. This set has a limit point 0 in $\mathbb{C}$, since $1/n \in \mathbb{Q}$, for $n = 1, 2, \ldots$, and $1/n \to 0$ as $n \to \infty$.

It follows from the Uniqueness Theorem that f_1 and f_2 agree throughout $\mathbb{C}$, so they are equal. 3

20 Total

Question 9

(a) (i) Observe that

$$\begin{aligned}
|\cosh z - 1| &= \left|\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right|\\
&\le \left|\frac{z^2}{2!}\right|+\left|\frac{z^4}{4!}\right|+\left|\frac{z^6}{6!}\right|+\cdots,
\end{aligned}$$

using the Triangle Inequality for Series. So if $|z| = 1$, then

$$|\cosh z - 1| \le \frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots = \cosh 1 - 1.$$

Now

$$\cosh 1 - 1 = \tfrac{1}{2}(e+e^{-1}) - 1 < \tfrac{1}{2}(3+1) - 1 = 1.$$

Hence $|\cosh z - 1| < 1$, for $|z| = 1$. 4

(ii) Let $f(z) = \cosh z - 1 - z$. We must find all solutions of the equation $f(z) = 0$ in the open unit disc $\{z : |z| < 1\}$. One solution is $z = 0$, because

$$f(0) = \cosh 0 - 1 - 0 = 0.$$

Next we define $g(z) = -z$. If $|z| = 1$, then

$$|f(z) - g(z)| = |\cosh z - 1| < 1,$$

by part (a)(i). Also, for $|z| = 1$, we have $|g(z)| = |z| = 1$. Hence

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } |z| = 1.$$

Now f and g are analytic on the simply connected region $\mathbb{C}$, and $\{z : |z| = 1\}$ is a simple-closed contour in $\mathbb{C}$, so we see from

Rouché's Theorem that f has the same number of zeros as g inside $\{z : |z| = 1\}$, namely 1.

It follows that $z = 0$ is the only solution of the equation $\cosh z = 1 + z$ in the open unit disc $\{z : |z| < 1\}$. 6

(b) Let $f(z) = z^2 \exp(1 + z^2)$ and $\mathcal{R} = \{z : |z| < 2\}$. Then f is analytic on $\mathbb{C}$, so it is analytic (and non-constant) on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}} = \{z : |z| \leq 2\}$. We can therefore apply the Maximum Principle to see that the maximum value of $|f(z)|$ on $\overline{\mathcal{R}}$ is attained on the boundary $\partial\mathcal{R}$ and is not attained in $\mathcal{R}$. Hence

$$\max\{|f(z)| : |z| \leq 2\} = \max\{|f(z)| : |z| = 2\}.$$

Now, if $|z| = 2$, then $z = 2e^{it}$, where $0 \leq t < 2\pi$. Hence

$$\begin{aligned}
|f(z)| &= |z^2 \exp(1 + z^2)| \\
&= 4|\exp(1 + 4e^{2it})| \\
&= 4|\exp(1 + 4\cos 2t + 4i \sin 2t)| \\
&= 4|\exp(1 + 4\cos 2t)|\,|\exp(4i \sin 2t)| \\
&= 4\exp(1 + 4\cos 2t).
\end{aligned}$$

Since $x \longmapsto e^x$ is an increasing real function, the expression $4\exp(1 + 4\cos 2t)$ takes its maximum value when $\cos 2t = 1$. This happens when (and only when) $t = 0, \pi$, corresponding to the values

$$z = 2e^{i0} = 2 \quad \text{and} \quad z = 2e^{i\pi} = -2.$$

At these values,

$$|f(z)| = 4\exp(1 + 4) = 4e^5.$$

In summary, then,

$$\max\{|z^2 \exp(1 + z^2)| : |z| \leq 2\} = 4e^5,$$

and this maximum is attained at the points $z = \pm 2$ only. 10

20 Total